Higher Signatures of Witt spaces

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Abstract

The signature is a fundamental homotopy invariant for oriented manifolds. However, for spaces with singularities, this usual notion of signature ceases to exist, since spaces with singularities fail the usual Poincaré duality in general. A generalized Poincaré duality theorem for spaces with singularities was proven by Goresky and MacPherson using intersection homology. The classical signature was then extended to Witt spaces by Siegel using this generalized Poincaré duality. In this paper, we study the higher signatures of Witt spaces by using noncommutative geometric methods.

1 Introduction

The signature is a fundamental invariant for oriented manifolds. The Hirzebruch signature theorem expresses the signature of an oriented manifold M in terms of characteristic classes:

$$\mathrm{sig}(M) = \langle \mathcal{L}(M), [M] \rangle \in \mathbb{Z},$$

where $\mathcal{L}(M) \in H^*(M;\mathbb{Q})$ is the $\mathcal{L}$-class of M, a certain power series in the Pontrjagin classes. Since the definition of the signature only depends on the cohomology ring of the manifold, it is clearly a homotopy invariant. Now suppose M is not simply connected with $\pi_1(M) = \Gamma$. Let $B\Gamma$ be the classifying space for Γ and $f : M \to B\Gamma$ be a continuous map. For each cohomology class $[x] \in H^*(B\Gamma;\mathbb{Q})$, one has the following characteristic number, called a higher signature:

$$\mathrm{sig}_{[x]}(M, f) = \langle \mathcal{L}(M) \cup f^*[x], [M] \rangle \in \mathbb{Q}.$$

The Novikov conjecture states that every higher signature is homotopy invariant, that is, for all orientation preserving homotopy equivalences $g : N \to M$ of closed oriented manifolds and all continuous maps $f : M \to B\Gamma$,

$$\mathrm{sig}_{[x]}(M, f) = \mathrm{sig}_{[x]}(N, g \circ f).$$

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This conjecture has been proved for a large class of groups [16, 11, 10, 25, 18, 19, 29, 30, 17, 9]. A common theme of the proofs for most of these cases, where the Novikov conjecture is known to hold, is to first show the so-called *strong Novikov conjecture* by using noncommutative geometry methods. Then the original Novikov conjecture follows as a consequence of the Strong Novikov conjecture. Recall that the strong Novikov conjecture says the following map, called the Baum-Connes assembly map,

$$\mu : K_i^{\Gamma}(\underline{E}\Gamma) \to K_i(C_r^*(\Gamma))$$

is injective, where $i = 0, 1$. Here $\underline{E}\Gamma$ is the universal space for proper Γ-actions, and $K_i^{\Gamma}(\underline{E}\Gamma)$ is the i-th Γ equivariant K-homology of $\underline{E}\Gamma$. Roughly speaking, every K-homology class in $K_i^{\Gamma}(\underline{E}\Gamma)$ can be represented by a Dirac type operator on some closed manifold. What the assembly map μ does is to map each of these Dirac type operators to its corresponding K-theoretical higher index.

When the assembly map is applied to the signature operator of a manifold, we call the resulting K-theoretical higher index the higher signature index class of the manifold. The higher signature index class is one of the most fundamental invariants for studying manifolds. In this paper we shall study a generalization of this invariant for a class of spaces with singularities, Witt spaces. The case where the fundamental group of the underlying Witt space is trivial has been studied by Siegel [24], based on the work of Goresky and MacPherson [12]. This case was also studied with an analytic approach by Cheeger [7, 8]. More recently, by generalizing Cheeger's work, Albin, Leichtnam, Mazzeo and Piazza used an analytic approach to study the higher signature index class for Witt spaces[2] (see also [3] for the higher signature index class of Cheeger spaces).

In this paper, we shall take a conceptual and combinatorial approach to the higher signature index class for Witt spaces, by using noncommutative geometric methods. Our approach is very much inspired by the work of Higson and Roe on mapping surgery exact sequence in topology to analytic exact sequence in K-theory [14, 15, 13]. The main methods of the paper are a combination of the techniques from the original approach of Goresky, MacPherson and Siegel [12] [24], and techniques from noncommutative geometry.

Here is a brief summary of the main results in the paper. Suppose X is a pseudomanifold (see Section 2.1). Let T be a triangulation of X. We denote the first barycentric subdivision of T by T'. Consider the stratification of X given by the skeleton of T,

$$X = |T_n| \supset \Sigma = |T_{n-2}| \supset |T_{n-3}| \supset \cdots \supset |T_0|.$$

Define $R_i^{\bar{p}}$ to be the subcomplex of T' consisting of all simplices which are $(\bar{p}, i)$-allowable with respect to this stratification, where $\bar{p}$ is a certain perversity (see Section 2.2). Let $W_i^{\bar{p}}(X)$ be the subgroup of $C_i^{T'}(R_i^{\bar{p}})$ consisting of those simplicial i-chains with boundary supported on $R_{i-1}^{\bar{p}}$. We define $W_{\bar{p}}^i(X) = \mathrm{Hom}_{fin}(W_i^{\bar{p}}(X), \mathbb{C})$ the group of finitely supported $(\bar{p}, i)$-allowable simplicial i-cochains. We denote the corresponding chain complex by $(W_*^{\bar{p}}(X), b)$ and $(W_{\bar{p}}^*(X), b^*)$ respectively. The following theorem states that, if X is an oriented Witt space, then X naturally gives rise a geometrically controlled Poincaré complex (see Theorem 3.14 below).

Theorem 1.1. *Every n-dimensional oriented Witt space X is a geometrically controlled Poincaré pseudomanifold of dimension n, that is, the duality chain map $\mathbb{P}: (W^*_{\bar{m}}(X), b^*) \to (W^{\bar{m}}_{n-*}(X), b)$ associated to the fundamental class $[X]$ is a chain equivalence in the geometrically controlled category. Here $\bar{m}$ is the lower middle perversity.*

We refer the reader to Section 2.5 and Definition 3.8 for the precise definitions of various terms.

The theorem above allows us to define the higher signature index class for Witt spaces. More precisely, Suppose X is a closed oriented Witt space of dimension n. Let $\widetilde{X}$ be a Γ-covering over X that is determined by a continuous map $f: X \to B\Gamma$. Here $B\Gamma$ is the classifying space of Γ. Consider the following analytically controlled Γ-equivariant Hilbert-Poincaré complex (see Section 2.4 and Section 4 for details):

$$E_0^{\bar{m}}(\widetilde{X}) \xleftarrow{b} E_1^{\bar{m}}(\widetilde{X}) \xleftarrow{b} \cdots \xleftarrow{b} E_n^{\bar{m}}(\widetilde{X}),$$

where $E_i^{\bar{m}}(\widetilde{X})$ is the Hilbert space completion of $W_i^{\bar{m}}(\widetilde{X})$. We denote the associated higher signature index class in $K_n(C_r^*(\Gamma))$ by $\mathrm{sig}_\Gamma(X, f)$ (cf. Section 2.4). Once casted in this framework, then the following invariance properties of the higher signature follow immediately from the general machinery for Hilbert-Poincaré complexes [13, Section 4 and Section 7].

Theorem 1.2. *(i) Higher signatures of Witt spaces are invariant under Witt cobordism. More precisely, suppose X_1 and X_2 are two closed oriented Witt spaces with continuous maps $f_1: X_1 \to B\Gamma$ and $f_2: X_2 \to B\Gamma$. If X_1 and X_2 are Γ-equivariantly cobordant, then*

$$\mathrm{sig}_\Gamma(X_1, f_1) = \mathrm{sig}_\Gamma(X_2, f_2)$$

in $K_n(C_r^*(\Gamma))$, where $n = \dim X_1 = \dim X_2$.

(ii) Higher signatures of Witt spaces are invariant under stratified homotopy equivalence. More precisely, X and Y are two closed oriented Witt spaces, and $f: Y \to B\Gamma$ is a continuous map. If $\varphi: X \to Y$ is a stratified homotopy equivalence, then

$$\mathrm{sig}_\Gamma(X, f \circ \varphi) = \mathrm{sig}_\Gamma(Y, f).$$

Here we refer to Definition 4.6 for the definition of stratified homotopy equivalence.

Now a natural question is to compare the higher signature index class defined in this paper with the (analytic) higher signature index class in the paper of Albin, Leichtnam, Mazzeo and Piazza [2]. In this case where the fundamental group is trivial, these two definitions of the signature index are clearly equivalent. It is reasonable to conjecture that the two definitions of the higher signature index class are always equivalent in general. One possible way to prove this is to build a natural chain isomorphism from the Poincaré complex of ℓ^2-differential forms (with certain constraints coming from the singularities) in [2] to the Hilbert-Poincaé complex in the current paper.

The paper is organized as follows. In Section 2, we recall various basic definitions and fix some notation. In Section 3, we associate naturally to each Witt space a

Poincaré complex of certain allowable simplicial chains. The main result of this section is to prove that such a construction is geometrically controlled. In Section 4, we define the higher signature index class for Witt spaces and prove its various invariance properties.

2 Preliminaries

In this section, let us recall some basic definitions and fix some notation.

2.1 Pseudomanifolds

In this paper, all spaces are assumed to be piecewise linear (abbreviated to p.l. from now on), unless otherwise specified. We first recall the definition of pseudomanifold.

Definition 2.1. (1) A pseudomanifold of dimension n is a locally compact space X containing a closed subspace Σ with $\dim(\Sigma) \leq n-2$ such that $X - \Sigma$ is an n-dimensional oriented manifold which is dense in X.

(2) A stratification of a pseudomanifold X is a filtration by closed subspaces

$$X = X_n \supset X_{n-1} = X_{n-2} = \Sigma \supset X_{n-3} \supset \cdots \supset X_1 \supset X_0$$

such that for each point $p \in X_i - X_{i-1}$ there is a filtered space

$$V = V_n \supset V_{n-1} \supset \cdots \supset V_i = \text{a point}$$

and a mapping $V \times B^i \to X$ which, for each j, takes $V_j \times B^i$ p.l. homeomorphically to a neighborhood of p in X_j. Here, B^i is the p.l. i-ball and p corresponds to the point $V_i \times$ (an interior point of B^i) .

If $X_i - X_{i-1}$ is nonempty, then it is a (usually open) manifold of dimension i, and is called the i-dimensional stratum of the stratification.

We also have the following definition of pseudomanifold with boundary.

Definition 2.2. An n-dimensional pseudomanifold with boundary is a pair of pseudomanifolds (X, Y) satisfying:

(1) Y is a pseudomanifold of dimension $(n-1)$ with singular set Σ_Y.

(2) Y is a closed subspace of X. If we denote the singular set of X by Σ_X, then $X - (\Sigma_X \cup Y)$ is an n-dimensional oriented manifold which is dense in X.

(3) Y is collared in X, that is, there is a closed neighborhood N of Y in X and an orientation preserving p.l. homeomorphism $Y \times [0,1] \to N$ which takes $\Sigma_Y \times [0,1]$ onto $\Sigma_X \cap N$.

Now we recall the definition of stratification for pseudomanifolds with boundary.

Definition 2.3. A stratification of an n-dimensional pseudomanifold with boundary (X, Y) is a filtration by closed subspaces

$$X = X_n \supset X_{n-1} = X_{n-2} = \Sigma \supset X_{n-3} \supset \cdots \supset X_1 \supset X_0$$

such that

(1) the filtration of Y given by $Y_{j-1} = X_j \cap Y$ stratifies Y;

(2) the filtration of $X - Y$ given by $X_j - Y_{j-1}$ stratifies X;

(3) these filtrations respect the collaring of Y in X, that is, the collaring homeomorphism takes $Y_{j-1} \times [0,1]$ to $X_j \cap N$.

2.2 Piecewise linear chains and intersection homology

Suppose X is an n-dimensional pseudomanifold with a fixed stratification. If T is a triangulation of X, let $C_*^T(X)$ be the chain complex of simplicial chains of X with respect to T.

Definition 2.4. The chain complex $C_*(X)$ of all piecewise linear geometric chains is the direct limit under refinement of $C_*^T(X)$, with T ranging over all triangulations of X compatible with the p.l. structure.

For $\xi \in C_i^T(X)$, the support of ξ, denoted by $|\xi|$, is the union of the closures of the i-simplices whose coefficients in ξ are not zero. The support of ξ is invariant under refinement. Therefore, a chain in $C_*(X)$ has a well-defined support.

Definition 2.5. A perversity is a sequence of integers $\bar{p} = (p_2, p_3, \cdots, p_n)$ such that $p_2 = 0$ and $p_{k+1} = p_k$ or $p_k + 1$.

The minimum perversity is given by $\bar{0} = (0, 0, \cdots, 0)$ and the maximum perversity is given by $\bar{t} = (0, 1, 2, \cdots, n-2)$.

Definition 2.6. Given an integer i and a perversity $\bar{p}$, a subspace $Y \subset X$ is called $(\bar{p}, i)$-allowable, if $\dim(Y) \leq i$ and $\dim(Y \cap X_{n-k}) \leq i - k + p_k$ for all $k \geq 2$.

The following lemma will be useful in later sections. Let us denote the i-dimensional stratum of the stratification by $\chi_i = X_i - X_{i-1}$.

Lemma 2.7 (cf. [23, Chapter I, Section 2])**.** *A subspace $Y \subset X$ is $(\bar{p}, i)$-allowable if and only if $\dim(Y) \leq i$ and $\dim(Y \cap \chi_{n-k}) \leq i - k + p_k$ for all $k \geq 2$.*

Definition 2.8. $C_i^{\bar{p}}(X)$ is the subgroup of $C_i(X)$ consisting of those chains ξ such that $|\xi|$ is $(\bar{p}, i)$-allowable and $|\partial\xi|$ is $(\bar{p}, i-1)$-allowable.

Now we are ready to define intersection homology of perversity $\bar{p}$.

Definition 2.9. The i-th intersection homology group of perversity $\bar{p}$, denoted $\mathrm{IH}_i^{\bar{p}}(X)$, is the i-th homology group of the chain complex $C_*^{\bar{p}}(X)$.

Recall that the fundamental class of a pseudomanifold is defined to be the unique class $[X] \in H_n(X)$ which restricts to the local orientation class in $H_n(X, X-p)$ for every $p \in X - \Sigma_X$. The Poincaré duality map is defined to be the cap product with this fundamental class

$$\frown [X] : H^{n-i}(X) \to H_i(X).$$

In general, this map is not necessarily an isomorphism, since X is singular. In any case, there are compatible homomorphisms

$$H^{n-i}(X) \xrightarrow{\alpha_{\bar{p}}} \mathrm{IH}_i^{\bar{p}}(X) \xrightarrow{\omega_{\bar{p}}} H_i(X)$$

which factor the Poincaré duality map [12, Section 1.4]. Here the map $\omega_{\bar{p}} : \mathrm{IH}_i^{\bar{p}}(X) \to H_i(X)$ is induced from the inclusions $C_*^{\bar{p}}(X) \subset C_*(X)$, and we will recall the detailed description of the map $\alpha_{\bar{p}} : H^{n-i}(X) \to \mathrm{IH}_i^{\bar{p}}(X)$ in Section 3.

Let $\bar{p}, \bar{q}$ and $\bar{r}$ be perversities such that $\bar{p}+\bar{q} \leq \bar{r}$. Then there is a unique intersection pairing [12, Section 2.3]:

$$\cap : \mathrm{IH}_i^{\bar{p}}(X) \times \mathrm{IH}_j^{\bar{q}}(X) \to \mathrm{IH}_{i+j-n}^{\bar{r}}(X).$$

We have the following fundamental theorem of intersection homology due to Goresky and MacPherson [12, Section 3.3].

Theorem 2.10 (Generalized Poincaré Duality)**.** *Let $\varepsilon : \mathrm{IH}_0^{\bar{t}}(X) \to \mathbb{Z}$ be the augmentation which counts points with multiplicity, where $\bar{t}$ is the maximal perversity $(0, 1, 2, \cdots, n-2)$. If $i + j = n$ and $\bar{p} + \bar{q} = \bar{t}$, then the augmented intersection pairing*

$$\mathrm{IH}_i^{\bar{p}}(X) \times \mathrm{IH}_j^{\bar{q}}(X) \to \mathrm{IH}_0^{\bar{t}}(X) \to \mathbb{Z}$$

is nondegenerate after tensoring with the rationals $\mathbb{Q}$.

2.3 Special simplicial chain complexes for Pseudomanifolds

In this paper, we need to work with chain complexes in the geometrically controlled category. In particular, instead of working with the chain complex $C_i^{\bar{p}}(X)$ above, it is more suitable for us to work with certain simplicial chain complexes. We recall these simplicial chain complexes in this subsection (cf. [12, Remark 3.4] and [23, Section 5 of Chapter 1]).

Given a pseudomanifold X, let T be a triangulation of X. We denote the first barycentric subdivision of T by T'. Consider the stratification of X given by the skeleton of T,

$$X = |T_n| \supset \Sigma = |T_{n-2}| \supset |T_{n-3}| \supset \cdots \supset |T_0|.$$

Define $R_i^{\bar{p}}$ to be the subcomplex of T' consisting of all simplices which are $(\bar{p}, i)$-allowable with respect to this stratification.

Now let $W_i^{\bar{p}}(X)$ be the subgroup of $C_i^{T'}(R_i^{\bar{p}})$ consisting of those simplicial i-chains with boundary supported on $R_{i-1}^{\bar{p}}$. In particular, $W_*^{\bar{p}}(X)$ is a subcomplex of $C_*^{\bar{p}}(X)$. Moreover, $W_*^{\bar{p}}(X)$ satisfies the following two properties:

(1) The inclusion $i : W_*^{\bar{p}}(X) \to C_*^{\bar{p}}(X)$ induces an isomorphism on homology.

(2) If $\bar{p} \leq \bar{q}$, then there is a natural inclusion of chain complexes $W_*^{\bar{p}} \hookrightarrow W_*^{\bar{q}}$.

2.4 Hilbert-Poincaré complexes

In this subsection, we recall the definition of Hilbert-Poincaré complexes, which are fundamental for studying higher signatures of topological spaces. We refer to [13] for more details.

Let A be a unital C^*-algebra. Consider a chain complex of Hilbert modules over A:

$$E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n$$

where the differentials b_j are bounded adjointable operators. The j-th homology of the complex is the quotient space obtained by dividing the kernel of b_j by the image of b_{j+1}. Note that, since the differentials need not to have closed range, the homology spaces are not necessarily Hilbert modules themselves.

Definition 2.11. An n-dimensional Hilbert-Poincaré complex (over a C^*-algebra A) is a complex of finitely generated Hilbert A-modules

$$E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n$$

together with adjointable operators $T : E_p \to E_{n-p}$ such that

(1) if $v \in E_p$, then $T^*v = (-1)^{(n-p)p}Tv$;

(2) if $v \in E_p$, then $Tb^*(v) + (-1)^p bT(v) = 0$;

(3) T induces an isomorphism from the homology of the dual complex

$$E_n \xleftarrow{b_n^*} E_{n-1} \xleftarrow{b_{n-1}^*} \cdots \xleftarrow{b_1^*} E_0$$

to the homology of the complex (E, b).

Now we will associate to each n-dimensional Hilbert-Poincaré complex an index class, called signature, in the K-theory group $K_n(A)$.

Definition 2.12. Let (E, b, T) be an n-dimensional Hilbert-Poincaré complex. We denote l to be the integer such that

$$n = \begin{cases} 2l & \text{if } n \text{ is even,} \\ 2l+1 & \text{if } n \text{ is odd.} \end{cases}$$

Define $S : E \to E$ to be the bounded adjointable operator such that

$$S(v) = i^{p(p-1)+l}T(v)$$

for $v \in E_p$. Here $i = \sqrt{-1}$.

It is not hard to verify that $S = S^*$ and $bS + Sb^* = 0$. Moreover, if we define $B = b + b^*$, then the self-adjoint operators $B \pm S : E \to E$ are invertible [13, Lemma 3.5].

Definition 2.13. (i) Let (E, b, T) be an odd-dimensional Hilbert-Poincaré complex. Its signature is the class in $K_1(A)$ of the invertible operator

$$(B + S)(B - S)^{-1} : E_{ev} \to E_{ev}$$

where $E_{ev} = \oplus_p E_{2p}$.

(ii) If (E, b, T) is an even-dimensional Hilbert-Poincaré complex, then its signature is the class in $K_0(A)$ determined by the formal difference $[P_+] - [P_-]$ of the positive projections of $B + S$ and $B - S$.

2.5 Geometric Hilbert-Poincaré complexes

In this subsection, we recall the definition of geometric Hilbert-Poincaré complexes. They are Hilbert-Poincaré complexes in the geometrically controlled category.

Definition 2.14. A simplicial complex X is of bounded geometry if there is a number N such that each of the vertices of X lies in at most N different simplices of X.

Denote by $C_*(X; \mathbb{C})$ the space of finitely supported simplicial chains on X, with complex coefficients. If no confusion arises, we shall write $C_*(X)$ instead of $C_*(X; \mathbb{C})$ for notational simplicity. Each vector space $C_k(X)$ has a natural basis, comprised of the k-simplices in X. With respect to this basis, we complete $C_k(X)$ into the Hilbert space $E_k(X)$ of square integrable simplicial k-chains on X. It is a X-module in the following natural way: if f is a continuous function vanishing at infinity and if $c = \sum c_\sigma[\sigma]$ is a square integrable k-chain, then we define

$$f \cdot c = \sum f(\hat{\sigma}) c_\sigma [\sigma]$$

where $\hat{\sigma}$ is the barycenter of σ. The simplicial differential $b : C_k(X) \to C_{k-1}(X)$ extends to a bounded operator on ℓ^2-chains, and we obtain a complex of Hilbert spaces

$$E_0(X) \xleftarrow{b_1} E_1(X) \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n(X).$$

We call this complex the ℓ^2-chain complex of X, and we call its adjoint

$$E_0(X) \xrightarrow{b_1^*} E_1(X) \xleftarrow{b_2^*} \cdots \xleftarrow{b_n^*} E_n(X)$$

the ℓ^2-cochain complex.

In the case where the simplicial complex X is given by a triangulation of an oriented manifold, the above ℓ^2-chain and ℓ^2-cochain complexes will naturally give rise to a Hilbert-Poincaré complex [14, Section 4]. However, since we are dealing with pseudomanifolds in this paper, we need an ℓ^2-version of the simplicial complex from Section 2.3.

Suppose X is an n-dimensional pseudomanifold. Given a perversity $\bar{p}$, let $W_*^{\bar{p}}(X)$ be the simplicial chain complex from Section 2.3. Fix a basis, consisting of minimal elements, for $W_k^{\bar{p}}(X)$, and complete it to the Hilbert space $E_k^{\bar{p}}(X)$ of square-integrable $(\bar{p}, k)$-allowable chains whose boundaries are square-integrable $(\bar{p}, k-1)$-allowable chain. Thanks to the bounded geometry of X, the simplicial differential

$b : W_k^{\bar{p}}(X) \to W_{k-1}^{\bar{p}}(X)$ extends to a bounded operator to $b : E_k^{\bar{p}}(X) \to E_{k-1}^{\bar{p}}(X)$. Given a pseudomanifold, the complexes

$$E_0^{\bar{p}}(X) \xleftarrow{b_1} E_1^{\bar{p}}(X) \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n^{\bar{p}}(X)$$

and

$$E_0^{\bar{p}}(X) \xrightarrow{b_1^*} E_1^{\bar{p}}(X) \xleftarrow{b_2^*} \cdots \xleftarrow{b_n^*} E_n^{\bar{p}}(X)$$

do *not* give rise to a Hilbert-Poincaré complex in general. We shall see later that, if X is a Witt space and $\bar{p} = \bar{m}$ the lower middle perversity, then the above complexes together with a natural Poincaré duality map define a Hilbert-Poincaré complex.

In order to have a suitable Poincaré duality map, we need to control our chain complexes and the maps between them in a geometric way.

Definition 2.15. Let X be a proper metric space. A complex vector space V is geometrically controlled over X if it is provided with a basis $B \subset V$ and a function $c : B \to X$ with the following property: for every $R > 0$, there is an $N < \infty$ such that if $S \subset X$ has diameter less than R then $c^{-1}(S)$ has cardinality less than N.

Definition 2.16. A linear map $T : V \to W$ is geometrically controlled over X if

(1) V and W are geometrically controlled;

(2) the matrix coefficients of T with respect to the given bases of V and W are uniformly bounded;

(3) and there is a constant $K > 0$ such that the (v, w)-matrix coefficients is zero whenever $d(c(v), c(w)) > K$.

Example 2.17. In this paper, the main example of a vector space geometrically controlled over X is the space $W_k^{\bar{p}}(X)$ of "allowable" simplicial chains of a triangulated pseudomanifold X. Here we always choose a basis $B = \{\mu_i\}$ consisting of minimal elements for $W_k^{\bar{p}}(X)$ (see Appendix A for details). Each minimal element μ_i of $W_k^{\bar{p}}(X)$ is supported on the star of some vertex v_{μ_i} in X. In particular, the function $c : B \to X$ is defined by $c(\mu_i) = v_{\mu_i}$. It is not difficult to see that the differential $b : W_k^{\bar{p}}(X) \to W_{k-1}^{\bar{p}}(X)$ is a geometrically controlled linear map. Similar remarks apply to the complex of finitely supported simplicial cochains.

3 Geometric Hilbert-Poincaré complexes associated to Witt spaces

In this section, we carry out a natural construction of geometric Hilbert-Poincaré complex for Witt spaces.

Given a pseudomanifold X, let T be a triangulation of X. Recall the construction of $W_i^{\bar{p}}$ from Section 2.3. Denote the first barycentric subdivision of T by T'. Consider the stratification of X given by the skeleton of T,

$$X = |T_n| \supset \Sigma = |T_{n-2}| \supset |T_{n-3}| \supset \cdots \supset |T_0|.$$

Define $R_i^{\bar{p}}$ to be the subcomplex of T' consisting of all simplices which are $(\bar{p}, i)$-allowable with respect to this stratification.

Definition 3.1. $W_i^{\bar{p}}(X)$ is defined to be the subgroup of $C_i^{T'}(R_i^{\bar{p}})$ consisting of those simplicial i-chains with boundary supported on $R_{i-1}^{\bar{p}}$.

Suppose $\dim X = n$, then we have the following chain complex:

$$W_0^{\bar{p}}(X) \xleftarrow{b} W_1^{\bar{p}}(X) \xleftarrow{b} \cdots \xleftarrow{b} W_n^{\bar{p}}(X).$$

Denote the group of finitely supported $(\bar{p}, i)$-allowable simplicial i-cochains by $W_{\bar{p}}^i(X) = \mathrm{Hom}_{fin}(W_i^{\bar{p}}(X), \mathbb{Z})$ and the corresponding cochain complex by

$$W_{\bar{p}}^n(X) \xleftarrow{b^*} W_{\bar{p}}^{n-1}(X) \xleftarrow{b^*} \cdots \xleftarrow{b^*} W_{\bar{p}}^0(X),$$

where b^* is the dual of b.

Now let T'' be the first barycentric subdivision of T' and let $\hat{\sigma}$ denote the barycenter of the simplex $\sigma \in T'$. Let T_i' be the i-skeleton of T', thought of as a subcomplex of T''. Observe that T_i' is spanned by all vertices $\hat{\sigma}$ such that $\dim(\sigma) \leq i$. Define the codimension i coskeleton D_i to be the subcomplex of T'' spanned by all vertices $\hat{\sigma}$ such that $\dim(\sigma) \geq i$. There are canonical simplicial deformation retracts:

$$X - |T_i'| \to |D_{i+1}| \quad \text{and} \quad X - |D_{i+1}| \to |T_i'|.$$

Indeed, T_i' and D_{i+1} are spanned by complementary sets of vertices. Each simplex in T'' is the join of its intersection with $|T_i'|$ and of its intersection with $|D_{i+1}|$. The deformation retracts are given by retractions along the join lines.

Recall the definition of the usual cap product from algebraic topology. Suppose φ is a finitely supported p-cochain, and $\sigma = [v_0, v_1, \cdots, v_{p+q}]$ is a $(p+q)$-simplex, then the cap product $\varphi \frown \sigma$ is the q-chain defined by

$$\varphi \frown \sigma = \varphi([v_p, \cdots, v_{p+q}])[v_0, \cdots, v_p];$$

we extend by linearity to a product between cochains and chains. Moreover, the cap product is related to the boundary and coboundary maps by the following standard formula

$$b(\varphi \frown \sigma) = \varphi \frown (b\sigma) - (-1)^{|\varphi|}(b^*\varphi) \frown \sigma$$

Remark 3.2. Although the formula of the (simplicial) cap product involves a choice of a partial ordering of all vertices (within a given orientation class), the cap product is independent of the partial ordering of vertices.

Let $C_i^{T'}(X)$ (resp. $C_i^{T''}(X)$) be the group of simplicial i-chains of T' (resp. T''). Recall that the inclusion $C_i^{T'}(X) \hookrightarrow C_i^{T''}(X)$ is a geometrically controlled chain homotopy equivalence. For a detailed proof, see for example [20, Theorem 17.2]. In particular, let us choose a chain homotopy inverse of this inclusion and denote it by $g_i : C_i^{T''}(X) \to C_i^{T'}(X)$.

Suppose $C^i_{T'}(X) = \mathrm{Hom}_{fin}(C_i^{T'}(X), \mathbb{Z})$ is the group of finitely supported i-cochains of T'. Let $[X]$ be the fundamental class of X. We define the following variant of the cap product map

$$\begin{tikzcd}
C^i_{T'}(X) = H^i_c(|T'_i|, |T'_{i-1}|) \arrow[d, "\frown [X]"] \\
H_{n-i}(X - |T'_{i-1}|, X - |T'_i|) \arrow[d, "\text{deformation retract}", "\cong"'] \\
H_{n-i}(|D_i|, |D_{i+1}|) \arrow[d, "\text{induced by inclusion}"] \\
H_{n-i}(|T''_{n-i}|, |T''_{n-i-1}|) = C^{T''}_{n-i}(X) \arrow[d, "g_i"] \\
C^{T'}_{n-i}(X)
\end{tikzcd}$$

where H^i_c stands for cohomology with compact support and g_i is a chain homotopy inverse of the inclusion $C_i^{T'}(X) \hookrightarrow C_i^{T''}(X)$.

Remark 3.3. The map $\frown [X]$ above is a simplicial version of the cap product defined by Whitehead [26].

We denote the composition of the above maps by

$$\mathbb{P} : C^i_{T'}(X) \to C^{T'}_{n-i}(X).$$

Notice that for any m-simplex $\xi \in T'$, we have $\dim(|D_i| \cap |\xi|) \leq m - i$. Therefore, $\dim(|D_i| \cap X_{n-k}) \leq n - i - k$. So all simplices in D_i are $(\bar{0}, n-i)$-allowable. Recall that

$$b(\varphi \frown [X]) = (-1)^{|\varphi|+1}(b^*\varphi) \frown [X]$$

for all simplicial cochains φ, where b and b^* are the boundary and coboundary maps. It follows that, for all $\varphi \in C^i_{T'}(X)$, we have $\mathbb{P}(\varphi) \in W^{\bar{\ell}}_{n-i}(X)$ for every perversity $\bar{\ell}$. In particular, $\mathbb{P}$ restricts to a map

$$\mathbb{P} : W^i_{\bar{p}}(X) \to W^{\bar{\ell}}_{n-i}(X)$$

for all perversities $\bar{p}$ and $\bar{\ell}$. Moreover, by construction, the map $\mathbb{P}$ is geometrically controlled.

From now on, we shall work with $W^i_{\bar{p}}(X) \otimes \mathbb{K}$ where $\mathbb{K} = \mathbb{Q}$ or $\mathbb{C}$. If no confusion arises, we write $W^i_{\bar{p}}(X)$ instead of $W^i_{\bar{p}}(X) \otimes \mathbb{K}$ for simplicity. Observe that, if we identify $W^i_{\bar{p}}(X)$ with $W^{\bar{p}}_i(X)$ by the canonical inner product, then the dual map of $\mathbb{P} : W^{n-i}_{\bar{\ell}}(X) \to W^{\bar{p}}_i(X)$ can be viewed as a map

$$\mathbb{P}^* : W^i_{\bar{p}}(X) \to W^{\bar{\ell}}_{n-i}(X).$$

To summarize, we have the following lemma.

Lemma 3.4. *Let X be an oriented pseudomanifold of dimension n and $[X]$ be the fundamental class of X. Then the map $\mathbb{P} : W^i_{\bar{p}}(X) \to W^{\bar{\ell}}_{n-i}(X)$ defined by*

$$\mathbb{P}\varphi = \varphi \frown [X]$$

is geometrically controlled and satisfies $b\mathbb{P}\varphi = (-1)^{|\varphi|+1}\mathbb{P}b^*\varphi$. Moreover, the geometrically controlled chain maps $\mathbb{P}$ and $(-1)^{|\varphi|(n-|\varphi|)}\mathbb{P}^*$ are chain homotopic (in the geometrically controlled category).

Proof. Only the last statement requires comment. The only essential difference between the chain maps $\mathbb{P}$ and $(-1)^{|\varphi|(n-|\varphi|)}\mathbb{P}^*$ is the different choices of partial ordering of vertices. Suppose we denote the partial ordering of vertices chosen for $\mathbb{P}$ by

$$\sigma = [v_0, \cdots, v_n]$$

for each simplex $\sigma \in [X]$. Then the partial ordering of vertices in the formula for $\mathbb{P}^*$ is given by $\bar{\sigma} = [v_n, v_{n-1}, \cdots, v_0]$ by reversing all the vertices for each $\sigma \in [X]$. To be more precise, since we need to preserve the orientation, a sign ought to be inserted. Let us define a map $\rho : C^{T'}_i(X) \to C^{T'}_i(X)$

$$\rho(\alpha) = \varepsilon_i \bar{\alpha}$$

for all $\alpha \in C^{T'}_i(X)$, where $\varepsilon_i = (-1)^{(i+1)i/2}$. It is not difficult to verify that

$$(-1)^{|\varphi|(n-|\varphi|)}\mathbb{P}^*(\varphi) = \varphi \frown (\rho[X]).$$

Recall that the definition of cap product does not depend on the partial ordering of vertices. Moreover, there is a geometrically controlled chain homotopy between $\frown [X]$ and $\frown (\rho[X])$. To see this, it suffices to show that reversing two consecutive vertices gives rise to a cap product that is chain homotopic to the original one. That is, the cap product formulas defined by the partial ordering $[v_0, \cdots, v_i, v_{i+1}, \cdots, v_n]$ and the partial ordering $(-1)[v_0, \cdots, v_{i+1}, v_i, \cdots, v_n]$ are chain homotopic (in the geometrically controlled category). This follows from a routine calculation . We leave the details to the reader.

□

Let $\bar{m} = (0, 0, 1, 1, 2, 2, \cdots)$ be the lower-middle perversity and $\bar{n} = (0, 1, 1, 2, 2, \cdots)$ be the upper-middle perversity.

Definition 3.5. Let X be an oriented pseudomanifold of dimension n. We say that X is a geometrically controlled Poincaré pseudomanifold of dimension n if a duality chain $\mathbb{P} : W^i_{\bar{m}}(X) \to W^{\bar{m}}_{n-i}(X)$ associated to the fundamental class $[X]$ is a chain equivalence in the geometrically controlled category.

Remark 3.6. Notice that the only perversity used in the above definition is the lower-middle perversity $\bar{m}$.

Suppose X is a geometrically controlled Poincaré pseudomanifold of dimension n. Complete the complexes $(W^*_{\bar{m}}(X), b^*)$ and $(W^{\bar{m}}_*(X), b)$ to complexes of Hilbert spaces, denoted by $(E^*_{\bar{m}}(X), b^*)$ and $(E^{\bar{m}}_*(X), b)$ respectively. Since $E^i_{\bar{m}}(X)$ and $E^{\bar{m}}_i(X)$ are identified via the canonical inner product, we shall only write $E^{\bar{m}}_i(X)$ from now on. The duality map $\mathbb{P}$ extends by continuity to a bounded operator P on $E^{\bar{m}}_i(X)$, and the operator

$$T = \frac{1}{2}(P + (-1)^{(n-i)i}P^*)$$

satisfies all three conditions in Definition 2.11. Therefore, the complex of Hilbert spaces

$$E^{\bar{m}}_0(X) \xleftarrow{b} E^{\bar{m}}_1(X) \xleftarrow{b} \cdots \xleftarrow{b} E^{\bar{m}}_n(X) \tag{1}$$

together with the duality operator T give rise to an analytically controlled Hilbert-Poincaré complex over X. We refer the reader to [13, Section 5] for the definition of analytically controlled complexes.

It remains to see for which pseudomanifolds the duality map $\mathbb{P} : W^i_{\bar{m}}(X) \to W^{\bar{m}}_{n-i}(X)$ is a chain equivalence. Let us first recall the following theorem of Goresky and MacPherson [12].

Theorem 3.7 (Generalized Poincaré duality [12]). *Let X be an oriented pseudomanifold of dimension n. If $\bar{p} + \bar{q} = \bar{t} = (0, 1, 2, \cdots, n-2)$, then the duality map $\mathbb{P} : W^i_{\bar{p}}(X) \to W^{\bar{q}}_{n-i}(X)$ is a chain equivalence (in the geometrically controlled category). In particular, the induced map on homology $\mathbb{P} : \mathrm{IH}^i_{\bar{p}}(X) \to \mathrm{IH}^{\bar{p}}_{n-i}(X)$ is an isomorphism for all $0 \leq i \leq n$.*

A special case of the above theorem is that

$$\mathbb{P} : (W^i_{\bar{m}}(X), b^*) \to (W^{\bar{n}}_{n-i}(X), b)$$

is a chain equivalence in the geometrically controlled category for all oriented pseudomanifold X of dimension n. Hence for a pseudomanifold to be geometrically controlled Poincaré pseudomanifold, it suffices to show that the canonical inclusion of chain complexes

$$\iota : (W^{\bar{m}}_*(X), b) \hookrightarrow (W^{\bar{n}}_*(X), b)$$

is a chain equivalence in the geometrically controlled category. In [24, Theorem 3.2 & 3.4], Siegel showed that the map $\iota : (W^{\bar{m}}_*(X), b) \hookrightarrow (W^{\bar{n}}_*(X), b)$ induces an isomorphism on homology for Witt spaces. In the following, we shall refine Siegel's argument to show that, for every Witt space, $\iota : (W^{\bar{m}}_*(X), b) \hookrightarrow (W^{\bar{n}}_*(X), b)$ is a chain equivalence in the geometrically controlled category.

Let us recall the definition of Witt spaces. Let X be n-dimensional pseudomanifold. For $x \in X$, the link of x, denoted by $lk(x, X)$ is unique up to p.l. homeomorphism [22]. Suppose $d(x)$ is the intrinsic dimension of X at x. Then there is a p.l. homeomorphism $lk(x, X) \cong S^{d(x)-1} * L(x)$, the join of the $(d(x)-1)$-dimensional sphere $S^{d(x)-1}$ and some pseudomanifold $L(x)$. The space $L(x)$ is of dimension $l(x) = n - d(x) - 1$, called the intrinsic link of x, which is unique up to p.l. homeomorphism (cf. [1]).

Definition 3.8. Let X be an n-dimensional pseudomanifold. We say X is a Witt space if

$$\mathrm{IH}^{\bar{m}}_{l(x)/2}(L(x);\mathbb{Q}) = 0$$

for all $x \in X$ such that $l(x) \equiv 0 \pmod 2$, that is, for all x in odd-codimensional stratum.

We have the following useful proposition due to Siegel.

Proposition 3.9 ([23, Chapter III, Proposition 2.6])**.** *Let X be a stratified pseudomanifold, with stratification*

$$X_n \supset X_{n-1} = X_{n-2} \supset \cdots X_0.$$

Let $L(\chi_i, x)$ be the link of $\chi_i = X_i - X_{i-1}$ at x. Then X is a Witt space if and only if

$$H^{\bar{m}}_{\ell}(L(\chi_i, x);\mathbb{Q}) = 0$$

for all $i = n-(2\ell+1)$ with $\ell \geq 1$.

Now we are ready to prove the main technical result in this section.

Proposition 3.10. *If X be a Witt space of dimension $n \geq 2$, then the canonical inclusion*

$$\iota : (W^{\bar{m}}_*(X), b) \hookrightarrow (W^{\bar{n}}_*(X), b)$$

is a chain equivalence in the geometrically controlled category.

Proof. We follow closely the Siegel's original argument [23, Chapter III, Section 3]. Let r be the largest integer such that $2r+1 \leq n$. We define $\bar{p}_k$ to be the perversity:

$$\bar{p}_k(c) = \begin{cases} \bar{m}(c) = [\frac{c-2}{2}] & \text{for } c \leq k \\ \bar{n}(c) = [\frac{c-1}{2}] & \text{for } c > k \end{cases}$$

where $1 \leq k \leq n$ and $2 \leq c \leq n$. Here $[s]$ stands for the largest integer that is less than or equal to s. Since $\bar{m}(c) = \bar{n}(c)$ for c even, we may assume k is odd. Note that $\bar{p}_{2r+1} = \bar{m}$ and $\bar{p}_1 = \bar{n}$. We have the following inclusions of chain complexes:

$$W^{\bar{m}}_* = W^{\bar{p}_{2r+1}}_* \subset W^{\bar{p}_{2r-1}}_* \subset \cdots \subset W^{\bar{p}_3}_* \subset W^{\bar{p}_1}_* = W^{\bar{n}}_*.$$

To prove the proposition, it suffices to show that

$$\iota : (W^{\bar{p}_{2s+3}}_*(X), b) \hookrightarrow (W^{\bar{p}_{2s+1}}_*(X), b)$$

is a chain equivalence for all $0 \leq s \leq r-1$. We shall prove this by constructing a chain homotopy $H_j : W^{\bar{p}_{2s+1}}_j \to W^{\bar{p}_{2s+1}}_{j+1}$ and a chain map $f_j : W^{\bar{p}_{2s+1}}_j \to W^{\bar{p}_{2s+1}}_j$ such that

(1) $bH + Hb = \mathbb{1} - f$, where $\mathbb{1}$ is the identity map;

(2) f_j restricts to $\mathbb{1}$ on $W^{\bar{p}_{2s+3}}_j$ and the image of f_j lies in $W^{\bar{p}_{2s+3}}_j$.

$$\begin{tikzcd}
\cdots & W_{j-1}^{\bar{p}_{2s+1}} \arrow[l] \arrow[d, "f_{j-1}"] \arrow[dr, "H_{j-1}"'] & W_j^{\bar{p}_{2s+1}} \arrow[l, "b_j"'] \arrow[d, "f_j"] \arrow[dr, "H_j"'] & W_{j+1}^{\bar{p}_{2s+1}} \arrow[l, "b_{j+1}"'] \arrow[d, "f_{j+1}"] & \cdots \arrow[l] \\
\cdots & W_{j-1}^{\bar{p}_{2s+1}} \arrow[l] & W_j^{\bar{p}_{2s+1}} \arrow[l, "b_j"] & W_{j+1}^{\bar{p}_{2s+1}} \arrow[l, "b_{j+1}"] & \cdots \arrow[l]
\end{tikzcd}$$

The constructions of the maps H and f are local. In particular, it will follow from construction that H and f are geometrically controlled.

Observe that $\bar{p}_{2s+1}$ and $\bar{p}_{2s+3}$ only differ at codimension $2s+3$. It follows that

$$W_j^{\bar{p}_{2s+3}} = W_j^{\bar{p}_{2s+1}}$$

for all $j \leq s+1$ and $j \geq n-s$. We define

$$f_j = \mathbb{1} \quad \text{and} \quad H_j = 0$$

for all $j \leq s+1$ and $j \geq n-s$. From now on, we assume that $s+1 < j < n-s$.

Recall that $\chi_{n-(2s+3)}$ is the $n-(2s+3)$ dimensional stratum. For each $z \in W_j^{\bar{p}_{2s+1}}$, we have

$$\dim(|z| \cap \chi_{n-(2s+3)}) \leq j - (2s+3) + \bar{p}_{2s+1}(2s+3) = j - s - 2.$$

If the following stronger inequality holds:

$$\dim(|z| \cap \chi_{n-(2s+3)}) \leq j - (2s+3) + \bar{p}_{2s+3}(2s+3) = j - s - 3,$$

then $z \in W_j^{\bar{p}_{2s+3}}$.

Let us fix a direct sum decomposition of $W_j^{\bar{p}_{2s+1}} = W_j^{\bar{p}_{2s+3}} \oplus V_j$ for each $s+1 < j < n-s$. We define $H_j(z) = 0$ and $f_j(z) = z$ for all $z \in W_j^{\bar{p}_{2s+3}}$. Let us fix a basis of minimal elements $\mathcal{B}$ for V_j. To define H_j and f_j, it suffices to define H_j and f_j for the basis elements in $\mathcal{B}$.

Note that $\chi_{n-(2s+3)}$ is the disjoint union of interiors of simplices $\sigma \in T$, where $\dim \sigma = n-(2s+3)$. Recall that here T is the chosen triangulation in the definition of $W_j^{\bar{p}}(X)$. For each $z \in \mathcal{B}$, let $\mathcal{C}_z$ be the nonempty set of simplices σ in T for which

$$\dim(|z| \cap \operatorname{Int}(\sigma)) = j - s - 2.$$

Now for each $\sigma \in \mathcal{C}_z$, consider the finite set $\{\tau_i\}_{i \in I}$ of $(j-s-2)$ simplices in the first barycentric subdivision of σ satisfying

$$\operatorname{Int}(\tau_i) \subset |z| \cap \operatorname{Int}(\sigma).$$

The subchain of z consisting of j-simplices in T' which intersect $\operatorname{Int}(\sigma)$ in $\operatorname{Int}(\tau_i)$ is:

$$z_i = \tau_i * v_i$$

where $v_i \in C_{s+1}^{T'}(\ell k(\sigma, T'))$, and $\ell k(\sigma, T')$ is the link of σ in T'.

We have the following lemma, which is a slight generalization of [23, Chapter III, Lemma 3.3].

Lemma 3.11. *$b(v_i) = 0$ for all i.*

Proof. Decompose z as $z = z_i + (z - z_i)$. Notice that

$$bz_i = b(\tau_i * v_i) = b\tau_i * v_i + (-1)^{j-s-1}\tau_i * bv_i.$$

Therefore, $|bz_i| \cap \mathrm{Int}(\sigma)$ contains $\mathrm{Int}(\tau_i)$ if and only if $bv_i \neq 0$.

Moreover, since $z \in W_j^{\bar{p}_{2s+1}}$, by definition $bz \in W_{j-1}^{\bar{p}_{2s+1}}$. In particular, it follows that

$$\dim(|bz| \cap \chi_{n-(2s+3)}) \leq j - 1 - (2s+3) + \bar{p}_{2s+1}(2s+3) = j - s - 3.$$

Therefore, $|bz| \cap \mathrm{Int}(\sigma)$ does not contain $\mathrm{Int}(\tau_i)$.

Now consider the decomposition

$$bz_i = -b(z - z_i) + bz.$$

By definition of z_i, we see that $|b(z - z_i)| \cap \mathrm{Int}(\sigma)$ does not contain $\mathrm{Int}(\tau_i)$.

Combining these observations, we see that $|bz_i| \cap \mathrm{Int}(\sigma)$ does not contains $\mathrm{Int}(\tau_i)$. Therefore, $bv_i = 0$. This finishes the proof. □

The canonical simplicial isomorphism of $\ell k(\sigma, T')$ and $\ell k(\sigma, T)'$ maps v_i to a cycle $\tilde{v}_i \in C_{s+1}^{T'}(\ell k(\sigma, T)')$. In particular, we have the complex $W_*^{\bar{m}}(\ell k(\sigma, T))$ associated to the restriction of T to $\ell k(\sigma, T)$. The proof of the following lemma can be found in [23, Chapter III, Lemma 3.4].

Lemma 3.12. *$\tilde{v}_i \in W_{s+1}^{\bar{m}}(\ell k(\sigma, T))$, for all i.*

By hypothesis, X is a Witt space. Proposition 3.9 implies that

$$\mathrm{IH}_{s+1}^{\bar{m}}(\ell k(\sigma, T)) = 0.$$

So there exists a chain $\tilde{x}_i \in W_{s+1}^{\bar{m}}(\ell k(\sigma, T))$ such that $b\tilde{x}_i = \tilde{v}_i$. Let x_i be the corresponding chain in $\ell k(\sigma, T')$. We define

$$w_i = (-1)^{j-s-1}\tau_i * x_i \quad \text{and} \quad w_\sigma = \sum_{i \in I} w_i.$$

Note that

$$bw_i = (-1)^{j-s-1} b\tau_i * x_i + \tau_i * v_i.$$

Lemma 3.13 ([23, Chapter III, Lemma 3.5])**.** *We have $w_\sigma \in W_{j+1}^{\bar{p}_{2s+1}}(X)$.*

Now repeat the argument for each of the simplices in $\mathcal{C}_z$. We define

$$H_j(z) = \sum_{\sigma \in \mathcal{C}_z} w_\sigma$$

The map $f_j : W_j^{\bar{p}_{2s+1}} \to W_j^{\bar{p}_{2s+1}}$ is defined by

$$f_j(z) := z - (b_{j+1} H_j(z) + H_{j-1} b_j(z)).$$

We shall verify that

(i) f_j restricts to $\mathbb{1}$ on $W_j^{\bar{p}_{2s+3}}$;

(ii) the image of f_j lies in $W_j^{\bar{p}_{2s+3}}$.

If $z \in W_j^{\bar{p}_{2s+3}}$, then $b_j(z) \in W_j^{\bar{p}_{2s+3}}$. Therefore, $H_j(z) = 0$ and $H_{j-1}(b_j(z)) = 0$ by definition. Hence follows part (i).

To see part (ii), we need to consider again the intersection of z with $\chi_{n-(2s+3)}$. Recall that $\mathcal{C}_z$ is the nonempty set of simplices σ in T for which

$$\dim(|z| \cap \mathrm{Int}(\sigma)) = j - s - 2.$$

Let $\mathcal{D}_z$ be the nonempty set of simplices σ in T for which

$$\dim(|z| \cap \mathrm{Int}(\sigma)) = (j-1) - s - 2.$$

Now for each $\sigma \in \mathcal{D}_z$, consider the finite set $\{\eta_k\}_{k\in K}$ of $(j-s-3)$ simplices in the first barycentric subdivision of σ satisfying

$$\mathrm{Int}(\eta_k) \subset |z| \cap \mathrm{Int}(\sigma).$$

The subchain of z consisting of j-simplices in T' which intersect $\mathrm{Int}(\sigma)$ in $\mathrm{Int}(\eta_k)$ is:

$$y_k = \eta_k * u_k$$

where $u_k \in C_{s+2}^{T'}(\ell k(\sigma, T'))$. In the construction for H_{j-1}, we see that the terms that are relevant for the definition of $H_{j-1}(bz)$ are

$$(b\tau_i) * v_i \text{ coming from } bz_i = (b\tau_i) * v_i,$$

and

$$(-1)^{j-s-2}\eta_k * (bu_k) \text{ coming from } by_k = (b\eta_k) * u_k + (-1)^{j-s-2}\eta_k * bu_k.$$

We define

$$w_i' = (-1)^{j-s-2}(b\tau_i) * x_i \quad \text{and} \quad w_\sigma' = \sum_{i\in I} w_i';$$

and

$$w_k'' = (-1)^{j-s-2}(-1)^{j-s-2}\eta_k * u_k = \eta_k * u_k \quad \text{and} \quad w_\sigma'' = \sum_{i\in I} w_k''.$$

It is not difficult to verify that

$$H_{j-1}(bz) = \sum_{\sigma\in\mathcal{C}_z} w_\sigma' + \sum_{\sigma\in\mathcal{D}_z} w_\sigma''.$$

Observe that $\dim(|H_{j-1}(bz)| \cap \chi_{n-(2s+3)}) \le j - s - 3$.

Recall that by the construction of w_σ, we have

$$\dim(|z - bw_\sigma| \cap \mathrm{Int}(\sigma)) \le j - s - 3,$$

hence it follows that

$$\dim(|z - bH_j(z)| \cap \chi_{n-(2s+3)}) \leq j - s - 3. \tag{2}$$

Since $f_j(z) = (z - bH_j(z)) - H_{j-1}b(z)$, we see that

$$\dim(|f_j(z)| \cap \chi_{n-(2s+3)}) \leq j - s - 3.$$

Moreover, notice that $bf_j(z) = (bz) - bH_{j-1}(bz)$. Apply formula (2) to bz. It follows immediately that

$$\dim(|bf_j(z)| \cap \chi_{n-(2s+3)}) \leq (j-1) - s - 3.$$

As a consequence, $f_j(z) \in W_j^{\bar{p}_{2s+3}}$. This proves part (ii).

Observe that the maps H_j and f_j are geometrically controlled by construction. This completes the proof of the proposition. □

To summarize, we have the following main theorem of this section.

Theorem 3.14. *Every n-dimensional oriented Witt space X is a geometrically controlled Poincaré pseudomanifold of dimension n, that is, the duality chain $\mathbb{P} : (W^*_{\bar{m}}(X), b^*) \to (W^{\bar{m}}_{n-*}(X), b)$ associated to the fundamental class $[X]$ is a chain equivalence in the geometrically controlled category.*

4 Higher signatures of Witt spaces

In this section, we define higher signatures for Witt spaces and prove some invariance properties.

Let us first recall some standard definitions from coarse geometry. We refer the reader to [21] [28] for more details. Let X be a proper metric space. That is, every closed ball in X is compact. An X-module is a separable Hilbert space equipped with a $*$-representation of $C_0(X)$, the algebra of all continuous functions on X which vanish at infinity. An X-module is called nondegenerate if the $*$-representation of $C_0(X)$ is nondegenerate. An X-module is said to be ample if no nonzero function in $C_0(X)$ acts as a compact operator.

Definition 4.1. Let H_X be a X-module and T a bounded linear operator acting on H_X.

(i) The propagation of T is defined to be $\sup\{d(x,y) \mid (x,y) \in \mathrm{Supp}(T)\}$, where $\mathrm{Supp}(T)$ is the complement (in $X \times X$) of the set of points $(x,y) \in X \times X$ for which there exist $f, g \in C_0(X)$ such that $gTf = 0$ and $f(x) \neq 0$, $g(y) \neq 0$;

(ii) T is said to be locally compact if fT and Tf are compact for all $f \in C_0(X)$.

Definition 4.2. Let H_X be an ample nondegenerate X-module and $B(H_X)$ the set of all bounded linear operators on H_X.

(i) The Roe algebra of X, denoted by $C^*(X)$, is the C^*-algebra generated by all locally compact operators with finite propagations in $B(H_X)$.

(ii) The localization algebra of X, denoted by $C_L^*(X)$, is the C^*-algebra generated by all bounded and uniformly continuous functions $f : [0, \infty) \to C^*(X)$ such that propagation of $f(t) \to 0$, as $t \to \infty$.

Now suppose $\widetilde{X}$ is a Γ-covering of X. Let $H_{\widetilde{X}}$ be a $\widetilde{X}$-module equipped with a covariant unitary representation of Γ. If we denote the representation of $C_0(\widetilde{X})$ by φ and the representation of Γ by π, this means

$$\pi(\gamma)(\varphi(f)v) = \varphi(f^\gamma)(\pi(\gamma)v),$$

where $f \in C_0(\widetilde{X})$, $\gamma \in \Gamma$, $v \in H_{\widetilde{X}}$ and $f^\gamma(x) = f(\gamma^{-1}x)$. In this case, we call $(H_{\widetilde{X}}, \Gamma, \varphi)$ a covariant system.

Definition 4.3. With the same notation above, we denote by $\mathbb{C}[\widetilde{X}]^\Gamma$ the $*$-algebra of all Γ-invariant locally compact operators with finite propagations in $B(H_{\widetilde{X}})$. We define $C^*(\widetilde{X})^\Gamma$ to be the completion of $\mathbb{C}[\widetilde{X}]^\Gamma$ in $B(H_{\widetilde{X}})$. The Γ-invariant version of the localization algebra, denoted by $C_L^*(\widetilde{X})^\Gamma$, is defined similarly.

If the action of Γ on $\widetilde{X}$ is cocompact, that is, if X is compact, then it is known that $C^*(\widetilde{X})^\Gamma$ is $*$-isomorphic to $C_r^*(\Gamma) \otimes \mathcal{K}$, where $C_r^*(\Gamma)$ is the reduced group C^*-algebra of Γ and $\mathcal{K}$ is the algebra of all compact operators. In particular, it follows that $K_i(C^*(\widetilde{X})^\Gamma) \cong K_i(C_r^*(\Gamma))$. Moreover, a theorem of Yu shows that there is a natural isomorphism $K_i(C_L^*(\widetilde{X})^\Gamma) \cong K_i^\Gamma(\widetilde{X}) = K_i(X)$ [28, Theorem 3.2].

Now suppose X is a closed oriented Witt space of dimension n. By Theorem 3.14 above, X is a geometrically controlled Poincaré pseudomanifold, that is, the duality chain $\mathbb{P} : (W_{\bar{m}}^*(X), b^*) \to (W_{n-*}^{\bar{m}}(X), b)$ associated to the fundamental class $[X]$ is a chain equivalence in the geometrically controlled category. By the discussion in Section 3, the duality map $\mathbb{P}$ extends by continuity to a bounded operator P on $E_i^{\bar{m}}(X)$, and the operator

$$T = \frac{1}{2}(P + (-1)^{(n-i)i} P^*)$$

satisfies all three conditions in Definition 2.11. Recall that $E_*^{\bar{m}}(X)$ is the Hilbert completion of $W_*^{\bar{m}}(X)$. Then the complex of Hilbert spaces

$$E_0^{\bar{m}}(X) \xleftarrow{b} E_1^{\bar{m}}(X) \xleftarrow{b} \cdots \xleftarrow{b} E_n^{\bar{m}}(X) \tag{3}$$

together with the duality operator T give rise to an analytically controlled Hilbert-Poincaré complex over X. We refer the reader to [13, Section 5] for the definition of analytically controlled complexes.

Now let $\widetilde{X}$ be the Γ-covering of X as above. Apply Theorem 3.14 to $\widetilde{X}$. We see that the duality chain $\mathbb{P} : (W_{\bar{m}}^*(\widetilde{X}), b^*) \to (W_{n-*}^{\bar{m}}(\widetilde{X}), b)$ associated to the fundamental class $[\widetilde{X}]$ is a chain equivalence in the category of geometrically controlled Γ-equivariant maps. Now a geometrically controlled Γ-equivariant Poincaré complex can be completed to yield an equivariantly analytically controlled Hilbert-Poincaré complex:

$$E_0^{\bar{m}}(\widetilde{X}) \xleftarrow{b} E_1^{\bar{m}}(\widetilde{X}) \xleftarrow{b} \cdots \xleftarrow{b} E_n^{\bar{m}}(\widetilde{X})$$

with the duality map $\widetilde{T}$ defined in the same as that of the unequivariant case above.

Before we define the notion of higher signature, let us briefly review the classical signature for Witt spaces in the current context. Recall the definition of signature for Hilbert-Poincaré complexes from Definition 2.13. In the case of an analytically controlled Hilbert-Poincaré complex

$$E_0^{\bar{m}}(X) \xleftarrow{b} E_1^{\bar{m}}(X) \xleftarrow{b} \cdots \xleftarrow{b} E_n^{\bar{m}}(X)$$

over a closed Witt space X of dimension n, its signature, denoted $\operatorname{sig}(X)$, lies in $K_n(\mathcal{K})$. Recall that $K_n(\mathcal{K}) = \mathbb{Z}$ when n is even and 0 when n is odd. It is not difficult to see that $\operatorname{sig}(X)$ agrees with the classical definition of signature for Witt spaces (cf. [13, Proposition 3.9]).

Remark 4.4. If X is a closed Witt space of dimension $4k$, then the classical signature of X is defined to be the signature of the symmetric bilinear form:

$$\cap : \mathrm{IH}_{2k}^{\bar{m}}(X) \times \mathrm{IH}_{2k}^{\bar{m}}(X) \to \mathbb{Q}.$$

Definition 4.5. Let X be a closed oriented Witt space of dimension n. Recall that every Γ-covering $\widetilde{X}$ of X is determined by a continuous map $f : X \to B\Gamma$, where $B\Gamma$ is the classifying space of Γ. The higher signature of $\widetilde{X}$ over X is defined to be the signature class of the analytically controlled Γ-equivariant Hilbert-Poincaré complex:

$$E_0^{\bar{m}}(\widetilde{X}) \xleftarrow{b} E_1^{\bar{m}}(\widetilde{X}) \xleftarrow{b} \cdots \xleftarrow{b} E_n^{\bar{m}}(\widetilde{X}).$$

This signature class is an element in $K_n(C_r^*(\Gamma) \otimes \mathcal{K}) = K_n(C_r^*(\Gamma))$ and is denoted by $\operatorname{sig}_\Gamma(X, f)$.

The higher signature of X, denoted by $\operatorname{sig}_\Gamma(X)$, is defined to the higher signature of the universal cover of X, where $\Gamma = \pi_1(X)$ in this case. Now we shall prove some natural invariance properties of higher signatures. Before we state the theorem, let us recall some standard definitions.

Definition 4.6. (1) Let X_1 and X_2 are two closed oriented Witt spaces with continuous maps $f_1 : X_1 \to B\Gamma$ and $f_2 : X_2 \to B\Gamma$. We say X_1 and X_2 are Γ-equivariantly cobordant if there exist a Witt space with boundary W and a continuous map $f : W \to B\Gamma$ such that $\partial W = X_1 \sqcup (-X_2)$, and $f|_{X_1} = f_1$ and $f|_{X_2} = f_2$.

(2) Suppose X and Y are two stratified spaces. A continuous map $\varphi : X \to Y$ is called stratum preserving if, for each stratum S of Y, the inverse image $\varphi^{-1}(S)$ is a union of strata of X. Such a stratum preserving map is called codimension preserving if, for each stratum of S of Y, we have

$$\operatorname{codim} \varphi^{-1}(S) = \operatorname{codim} S.$$

A stratified homotopy equivalence between X and Y is a homotopy equivalence in the category of codimension preserving maps.

Theorem 4.7. *(i) Higher signatures of Witt spaces are invariant under Witt cobordism. More precisely, suppose X_1 and X_2 are two closed oriented Witt spaces with continuous maps $f_1 : X_1 \to B\Gamma$ and $f_2 : X_2 \to B\Gamma$. If X_1 and X_2 are Γ-equivariantly cobordant, then*

$$\mathrm{sig}_\Gamma(X_1, f_1) = \mathrm{sig}_\Gamma(X_2, f_2)$$

in $K_n(C_r^*(\Gamma))$, where $n = \dim X_1 = \dim X_2$.

(ii) Higher signatures of Witt spaces are invariant under stratified homotopy equivalence. More precisely, X and Y are two closed oriented Witt spaces, and $f : Y \to B\Gamma$ is a continuous map. If $\varphi : X \to Y$ is a stratified homotopy equivalence, then

$$\mathrm{sig}_\Gamma(X, f \circ \varphi) = \mathrm{sig}_\Gamma(Y, f).$$

Proof. (i) Let W be a Witt cobordism between X_1 and X_2. Then the pair $(W, X_1 \sqcup (-X_2))$ together with the relevant maps to $B\Gamma$ give rise to a geometrically controlled Poincaré pair in the sense of [14, Section 3.2]. Now the statement follows immediately from [13, Theorem 7.6].

(ii) Without loss of generality, we assume φ is simplicial with respect to some triangulations of X and Y. The map φ induces a geometrically controlled homotopy equivalence between the associated geometrically controlled Poincaré complexes of X and Y. Now the statement follows immediately from [13, Theorem 4.3].

□

5 K-homology classes of signature operators

In this section, for each closed oriented Witt space X, we shall construct the K-homology class of its signature operator. The image of this K-homology class under the Baum-Connes assembly map is the higher signature of X from the previous section.

Recall that the Baum-Connes assembly map takes each K-homology class of X to its higher index:

$$\mu : K_i^\Gamma(X) \to K_i(C_r^*(\Gamma)).$$

In the case where $X = \underline{E}\Gamma$ is the universal space for Γ-proper actions, the Baum-Connes conjecture states that μ is an isomorphism [4][5].

Recall that we have $K_i(C_L^*(X)) \cong K_i(X)$, where $C_L^*(X)$ is the localization algebra of X (see Definition 4.2). To construct the K-homology class of the signature operator, we shall construct its corresponding element in $K_n(C_L^*(X))$, where $n = \dim X$.

Recall that in the construction of the localization algebra $C_L^*(X)$ of X, we need to choose a nondegenerate X-module. In this section, we shall fix an explicit choice of nondegenerate X-module that suits best with the purpose of our constructions.

Let X be closed oriented Witt space of dimension n. Recall the construction of $W_i^{\bar{m}}(X)$ from Section 2.3. Here $\bar{m}$ is the lower middle perversity. Denote the first barycentric subdivision of T by T'. Consider the stratification of X given by the skeleton of T,

$$X = |T_n| \supset \Sigma = |T_{n-2}| \supset |T_{n-3}| \supset \cdots \supset |T_0|.$$

Define $R_i^{\bar{m}}$ to be the subcomplex of T' consisting of all simplices which are $(\bar{p}, i)$-allowable with respect to this stratification. $W_i^{\bar{p}}(X)$ is defined to be the subgroup of $C_i^{T'}(R_i^{\bar{p}})$ consisting of those simplicial i-chains with boundary supported on $R_{i-1}^{\bar{p}}$.

Let us define $H_0 = \bigoplus_k W_k^{\bar{m}}(X) \otimes \mathbb{C}$. Fix a basis $B_k = \{\mu_i^k\}$ consisting of minimal elements for $W_k^{\bar{p}}(X)$ (see Appendix A for details). Each minimal element μ_i^k of $W_k^{\bar{p}}(X)$ is supported on the star of some vertex v_i^k in X. We define an X-module structure on H_0 as follows. Let every function $f \in C(X)$ act on H_0 by

$$f \cdot \mu_i^k = f(v_i^k)\mu_i^k.$$

5.1 Refinement map

In this subsection, let us describe a refinement procedure for a given triangulation T. This refinement procedure produces a particular subdivision of T, denoted by $\mathcal{A}T$ such that all successive refinements $\mathcal{A}^n T$ have bounded geometry, where the bound is uniform respect to $n \in \mathbb{N}$.

Let us recall the notion of typed simplicial complexes (cf. [6] [19]).

Definition 5.1. Suppose X is a simplicial complex of dimension n. Let X^0 be the set of vertices of X. A *type* on X is a map $\varphi : X^0 \to \{0, 1, \cdots, n\}$ such that for any simplex $\omega \in X$, the images by φ of the vertices of ω are pairwise distinct. A simplicial complex equipped with a type is said to be *typed*.

Given any simplicial complex X of dimension n, we denote its barycentric subdivision by Y. Then Y admits a type. Indeed, Y is the set of totally ordered subsets of X, that is,

$$Y^k = \{(\sigma_0, \cdots, \sigma_k) \mid \sigma_j \in X \text{ and } \sigma_i \text{ is a face of } \sigma_{i+1}\}.$$

Now the dimension function, which maps each barycenter of a simplex of X to the dimension of that simplex, is a type on Y.

Now suppose X is a typed simplicial complex of dimension n. In particular, this gives a consistent way of ordering the vertices of each simplex in X according to the type map. Therefore, each k-simplex of X can be canonically identified with the standard k-simplex Δ^k. Now to define our refinement procedure, it suffices to describe certain subdivisions of the standard simplices so that the number of simplices containing a vertex remains uniformly bounded for all successive subvisions. One way to achieve this is by the so-called standard subdivision [27, Appendix II.4]. In below, we briefly recall the construction of standard subdivision, and refer the reader to [27, Appendix II.4] for more details.

Let $\sigma = [v_0, v_1, \cdots, v_k]$ be a standard simplex with its vertices given in the order shown. Set

$$v_{ij} = \frac{1}{2}v_i + \frac{1}{2}v_j, \quad i \leq j;$$

in particular, $v_{ii} = v_i$. These are the vertices of the standard subdivision of σ, denoted $\mathcal{S}\sigma$. Define a partial ordering on these vertices by setting

$$v_{ij} \leq v_{kl} \quad \text{if} \quad k \leq i \text{ and } j \leq l.$$

Now the simplices of $\mathcal{S}\sigma$ are all those formed from the v_{ij} which are in increasing order. Moreover, each simplex in $\mathcal{S}\sigma$ naturally inherits an ordering of vertices from the above partial ordering of v_{ij}. It is not difficult to verify that $\mathcal{S}\sigma$ carries a natural type by mapping $v_{ij} \mapsto (j - i)$.

To summarize, given a typed simplicial complex X of dimension n, we apply the above standard subdivision procedure (consistently) to each n-simplex of X. We call the resulting simplicial complex the standard subdivision of X, denoted by $\mathcal{S}X$. Note that $\mathcal{S}X$ is also typed.

5.2 K-homology class

Let $\mathcal{A}$ be the refinement map from above and $\mathcal{A}T'$ be the resulting refinement of the triangulation T'. Moreover, the corresponding group $W_i^{\bar{p}}(X)$ with respect to $\mathcal{A}T'$ will be denoted by $W_i^{\bar{p}}(\mathcal{A}X)$. We define

$$H_1 = \bigoplus_k W_k^{\bar{m}}(\mathcal{A}X) \otimes \mathbb{C}.$$

Similarly, we fix a basis of minimal elements of H_1, and endow H_1 with an X-module structure accordingly.

Repeat the above process, and define

$$H_j = \bigoplus_k W_k^{\bar{m}}(\mathcal{A}^j X) \otimes \mathbb{C}.$$

Define H to be the ℓ^2-completion of $\bigoplus_{j=0}^{\infty} H_j$. Then H inherits an X-module structure from those of H_k. Moreover, it is not difficult to see H is an ample nondegenerate X-module.

By Theorem 3.14, each H_k, together with the maps b, b^* and T, gives rises to a geometric Hilbert-Poincaré complex of X. Let B_k and S_k be the operators on H_k corresponding to the operators B and S from Section 2.4. By construction, B_k and S_k have finite propagation. Moreover, $B_k \pm S_k$ are invertible for each $k \geq 0$. The following lemma shows that in fact $B_k \pm S_k$ are uniformly bounded below for all $k \geq 0$.

Lemma 5.2. *There exist constants $\varepsilon > 0$ and $C > 0$ such that*

$$\varepsilon < \|B_k \pm S_k\|_{H_k} < C$$

for all k.

Proof. Consider the disjoint union of countably many copies of X, denoted by $\coprod_j X_j$, where $X_j = X$ endowed with the triangulation $\mathcal{A}^j T'$. The stratification of each X_j remains the same, and is given by the skeleton of T:

$$X = |T_n| \supset \Sigma = |T_{n-2}| \supset |T_{n-3}| \supset \cdots \supset |T_0|.$$

Note that $\coprod_j X_j$ has bounded geometry under the above simplicial structure. Let $\{W_k^{\bar{m}}(\mathcal{A}^j X)\}$ be the corresponding geometric Hilbert-Poincaré complex over X_k. Then $\coprod_j H_j$ is a geometric Hilbert-Pincaré complex over $\coprod_j X_j$. Let $\mathcal{H}$ be the ℓ^2 completion of $\coprod_j H_j$. Then by the discussion in Section 2.5, the operators

$$\coprod_j (B_j + S_j) \text{ and } \coprod_j (B_j - S_j)$$

are bounded and invertible [13, Lemma 3.5]. In particular, there exist $\varepsilon > 0$ and C such that

$$\varepsilon < \|\coprod_j (B_j + S_j)\|_{\mathcal{H}} < C \text{ and } \varepsilon < \|\coprod_j (B_j - S_j)\|_{\mathcal{H}} < C.$$

It follows that $\varepsilon < \|B_j \pm S_j\|_{H_j} < C$ for all $j \geq 0$. □

5.2.1 Odd case

Let $p(x)$ be a polynomial on $[\varepsilon, C] \cup [-C, -\varepsilon]$ such that

$$\sup_{x \in [\varepsilon, C]} |f(x) - p(x)| < \frac{1}{C}.$$

Then $\|p(B_j - S_j) - (B_j - S_j)^{-1}\| < \frac{1}{\|B_j - S_j\|}$, which implies that $p(B_j - S_j)$ is invertible. Moreover, the element

$$(B_j + S_j) \cdot p(B_j - S_j)$$

has finite propagation. Since the propagation of $B_j - S_j$ goes to zero as j goes to ∞, we have that the propagation of $(B_j + S_j) \cdot p(B_j - S_j)$ goes to zero, as j goes to infinity.

The refinement map $\mathcal{A}$ induces a controlled chain homotopy equivalence

$$\mathcal{A}_j : (H_j, b) \to (H_{j+1}, b).$$

Observe that the propagation of $\mathcal{A}_j$ goes to zero, as $j \to \infty$. Moreover, $\mathcal{A}_j S_j \mathcal{A}_j^*$ is controlled chain homotopic to S_{j+1}. We shall use these controlled chain homotopy equivalences to construct a norm-bounded and uniformly continuous path that connects all $(B_j + S_j) \cdot p(B_j - S_j)$. The resulting path represents a class in $K_1(C_L^*(X))$, which is precisely the K-homology class of the signature operator of X.

Consider the duality operator $(-S_j) \oplus S_{j+1}$ on the chain complex $H_j \oplus H_{j+1}$. We shall construct a continuous path of invertible elements (with controlled propagations) connecting

$$\left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} + \begin{pmatrix} -S_j & \\ & S_{j+1} \end{pmatrix}\right] \cdot p\left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} - \begin{pmatrix} -S_j & \\ & S_{j+1} \end{pmatrix}\right]$$

to the identity operator $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The construction is adapted from [13, Section 4]. First consider the path of duality operators

$$\begin{pmatrix} -S_j & 0 \\ 0 & (1-t)S_{j+1} + t\mathcal{A}_j S_j \mathcal{A}_j^* \end{pmatrix}$$

which connects the duality operator $(-S_j) \oplus S_{j+1}$ on the chain complex $H_j \oplus H_{j+1}$ to the operator $(-S_j) \oplus \mathcal{A}_j S_j \mathcal{A}_j^*$. Following this, the path of duality operators

$$\begin{pmatrix} -\cos(t) S_j & \sin(t) S_j \mathcal{A}_j^* \\ \sin(t) \mathcal{A}_j S_j & \cos(t) \mathcal{A}_j S_j \mathcal{A}_j^* \end{pmatrix}, \ t \in [0, \pi/2],$$

connecting $(-S_j) \oplus \mathcal{A}_j S_j \mathcal{A}_j^*$ to $\left(\begin{smallmatrix} 0 & S_j \mathcal{A}_j^* \\ \mathcal{A}_j S_j & 0 \end{smallmatrix}\right)$.

By using these paths of duality operators, we see that

$$\left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} + \begin{pmatrix} -S_j & \\ & S_{j+1} \end{pmatrix}\right] \cdot p \left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} - \begin{pmatrix} -S_j & \\ & S_{j+1} \end{pmatrix}\right]$$

is connected to

$$\left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} + \begin{pmatrix} 0 & S_j \mathcal{A}_j^* \\ \mathcal{A}_j S_j & 0 \end{pmatrix}\right] \cdot p \left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} - \begin{pmatrix} 0 & S_j \mathcal{A}_j^* \\ \mathcal{A}_j S_j & 0 \end{pmatrix}\right]$$

by a norm-continuous path of invertibles.

Now observe that the duality operator $\left(\begin{smallmatrix} 0 & S_j \mathcal{A}_j^* \\ \mathcal{A}_j S_j & 0 \end{smallmatrix}\right)$ is connected to its additive inverse $\left(\begin{smallmatrix} 0 & -S_j \mathcal{A}_j^* \\ -\mathcal{A}_j S_j & 0 \end{smallmatrix}\right)$ by the path of duality operators

$$\begin{pmatrix} 0 & \exp(it) S_j \mathcal{A}_j^* \\ \exp(-it) \mathcal{A}_j S_j & 0 \end{pmatrix}, \ t \in [0, \pi].$$

To proceed, we need the following lemma.

Lemma 5.3. *The elements*

$$\mathcal{E}_j^{\pm}(t) = \begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} \pm \begin{pmatrix} -S_j & 0 \\ 0 & (1-t) S_{j+1} + t \mathcal{A}_j S_j \mathcal{A}_j^* \end{pmatrix},$$

$$\mathcal{F}_j^{\pm}(t) = \begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} \pm \begin{pmatrix} -\cos(t) S_j & \sin(t) S_j \mathcal{A}_j^* \\ \sin(t) \mathcal{A}_j S_j & \cos(t) \mathcal{A}_j S_j \mathcal{A}_j^* \end{pmatrix}$$

and

$$\mathcal{G}_j^{\pm}(t) = \begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} \pm \begin{pmatrix} 0 & \exp(it) S_j \mathcal{A}_j^* \\ \exp(-it) \mathcal{A}_j S_j & 0 \end{pmatrix}$$

are invertible. Moreover, there exists a constant ε and C such that

$$\varepsilon \leq \|\mathcal{E}_j^{\pm}(t)\|, \|\mathcal{F}_j^{\pm}(t)\|, \|\mathcal{G}_j^{\pm}(t)\| \leq C$$

for all j and t.

Proof. The proof uses the same idea from Lemma 5.2.

Consider the map

$$\mathcal{A} = \coprod_{j \geq 0} \mathcal{A}_j : \coprod_{j \geq 0} X_j \to \coprod_{j \geq 1} X_j$$

defined by $\mathcal{A}|_{X_j} = \mathcal{A}_j$. Similarly, we define

$$S = \coprod_{j\geq 0} S_j \text{ and } S' = \coprod_{j\geq 1} S_j;$$

$$B = \coprod_{j\geq 0} B_j \text{ and } B' = \coprod_{j\geq 1} B_j;$$

We have that $\mathcal{A}S\mathcal{A}^*$ is controlled chain homotopic to S'. Now define the paths of operators

$$\mathcal{E}^+(t) = \begin{pmatrix} B & \\ & B' \end{pmatrix} + \begin{pmatrix} -S & 0 \\ 0 & (1-t)S' + t\mathcal{A}S\mathcal{A}^* \end{pmatrix}.$$

By the discussion in Section 2.5, the operators $\mathcal{E}^+(t)$ are bounded and invertible [13, Lemma 3.5]. Therefore there exists a constant ε and C such that

$$\varepsilon \leq \|\mathcal{E}^+(t)\| \leq C.$$

The same argument applies to the other terms. Since there are only a finite number of paths, this finishes the proof. □

Without loss of generality, we assume that we have chosen ε and C as in the above lemma. It follows that the element

$$v_0 = \left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} + \begin{pmatrix} 0 & S_j\mathcal{A}_j^* \\ \mathcal{A}_jS_j & 0 \end{pmatrix}\right] \cdot p\left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} - \begin{pmatrix} 0 & S_j\mathcal{A}_j^* \\ \mathcal{A}_jS_j & 0 \end{pmatrix}\right]$$

is connected to

$$v_1 = \left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} + \begin{pmatrix} 0 & S_j\mathcal{A}_j^* \\ \mathcal{A}_jS_j & 0 \end{pmatrix}\right] \cdot p\left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} + \begin{pmatrix} 0 & S_j\mathcal{A}_j^* \\ \mathcal{A}_jS_j & 0 \end{pmatrix}\right]$$

by the path

$$v_t = \left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} + \begin{pmatrix} 0 & S_j\mathcal{A}_j^* \\ \mathcal{A}_jS_j & 0 \end{pmatrix}\right] \cdot p\left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} - \begin{pmatrix} 0 & \exp(it)S_j\mathcal{A}_j^* \\ \exp(-it)\mathcal{A}_jS_j & 0 \end{pmatrix}\right].$$

Notice that, since $p(x)$ is approximating the function $f(x) = x^{-1}$, the element v_1 in this path is very close to the identity operator $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. More precisely, the linear path between v_1 and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is a path of invertible elements connecting v_1 and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

To summarize, we have obtained a norm-continuous path of invertible elements that connects

$$\begin{aligned} &\left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} + \begin{pmatrix} -S_j & \\ & S_{j+1} \end{pmatrix}\right] \cdot p\left[\begin{pmatrix} B_j & \\ & B_{j+1} \end{pmatrix} - \begin{pmatrix} -S_j & \\ & S_{j+1} \end{pmatrix}\right] \\ &= \begin{pmatrix} (B_j - S_j)\cdot p(B_j+S_j) & 0 \\ 0 & (B_{j+1}+S_{j+1})\cdot p(B_{j+1}-S_{j+1}) \end{pmatrix} \end{aligned}$$

to the identity operator $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In particular, by multiplying this path by the element

$$\begin{pmatrix} (B_j + S_j)\cdot p(B_j - S_j) & 0 \\ 0 & 1 \end{pmatrix},$$

we have a path of invertible elements connecting $\left(\begin{smallmatrix}(B_j+S_j)\cdot p(B_j-S_j) & 0\\ 0 & 1\end{smallmatrix}\right)$ to

$$\left(\begin{smallmatrix}(B_j+S_j)\cdot p(B_j-S_j)(B_j-S_j)\cdot p(B_j+S_j) & 0\\ 0 & (B_{j+1}+S_{j+1})\cdot p(B_{j+1}-S_{j+1})\end{smallmatrix}\right).$$

Observe that again the entry

$$(B_j + S_j) \cdot p(B_j - S_j)(B_j - S_j) \cdot p(B_j + S_j)$$

in the last element is connected to the identity operator by a linear path of invertible elements. Therefore, combining these paths together, we obtain a path of invertible elements, denoted by U_t, $t \in [j, j+1]$, connecting

$$U_j = \begin{pmatrix}(B_j + S_j) \cdot p(B_j - S_j) & 0\\ 0 & 1\end{pmatrix}$$

to

$$U_{j+1} = \begin{pmatrix}1 & 0\\ 0 & (B_{j+1} + S_{j+1}) \cdot p(B_{j+1} - S_{j+1})\end{pmatrix}.$$

Let c_j be the maximum of the propagations of B_j, S_j and $\mathcal{A}_j$. By construction, the propagation of U_t is uniformly bounded by $N \cdot c_j$ for all $t \in [j, j+1]$. Here N is a universal constant that only depends on the degree of the polynomial $p(x)$.

Now view each U_t, $t \in [j, j+1]$, as an invertible operator on H by making it act as the identity operator on H_i for $i \neq j, j+1$. By putting all the intervals $[j, j+1]$ together, we obtain a bounded uniformly continuous path of invertible elements

$$U : [0, \infty) \to C^*(X)^+$$

such that the propagation of U_t goes to zero, as $t \to \infty$. Here $C^*(X)^+$ is the unitization of $C^*(X)$.

Definition 5.4. The K-homology class of the signature operator of X is defined to be the K-theory class of the path U in $K_1(C_L^*(X))$. Let us denote this K-homology class by $[D_{\mathrm{sig}}]$ from now on.

5.2.2 Even case

The even case is similar. We shall only point out a few key points and skip most of the details. Recall that the signature class of X is defined to be the element in $K_0(C^*(X))$ determined by the formal difference $[P_+] - [P_-]$ of the positive projections of $B_j + S_j$ and $B_j - S_j$. In other words, the signature class of X is the formal difference

$$g(B_j + S_j) - g(B_j - S_j)$$

where $g(x)$ is the function on $[\varepsilon, C] \cup [-C, -\varepsilon]$ such that $g(x) \equiv 1$ on $[\varepsilon, C]$ and $g(x) \equiv 0$ on $[-C, -\varepsilon]$. In order to have good control over the propagation, we approximate g sufficiently well by a polynomial $h(x)$ on $[\varepsilon, C] \cup [-C, -\varepsilon]$. Notice that, $h^2(x) - h(x) \neq 0$

in general. As a result, $h(B_j \pm S_j)$ are not projections. However, by choosing h sufficiently close to g, we have

$$\|h(B_j \pm S_j)^2 - h(B_j \pm S_j)\| < \delta$$

with δ sufficiently small. In other words, $h(B_j \pm S_j)$ are δ-quasi-projections. Now proceed as in the odd case (with obvious modifications), we obtain a bounded uniformly continuous path of δ-quasi-projections

$$Q : [0, \infty) \to C^*(X)$$

such that the propagation of Q_t goes to zero, as $t \to \infty$. In particular, Q gives rise to a δ-quasi-projection in $C_L^*(X)$, which in turn determines a K-theory class in $K_0(C_L^*(X))$ by the standard holomorphic functional calculus.

Definition 5.5. The K-homology class of the signature operator on X is defined to be the K-theory class in $K_0(C_L^*(X))$ determined by Q. Again we denote this K-homology class by $[D_{\mathrm{sig}}]$.

5.2.3 Equivariant case

The case of coverings of Witt spaces is completely similar. Suppose $\widetilde{X}$ is a Γ-covering of X, where Γ is a discrete group. Our primary example is $\widetilde{X}$ is the universal cover of X and $\Gamma = \pi_1(X)$.

We proceed exactly the same as the non-equivariant case above, but this time on the space $\widetilde{X}$. Notice that all operators are invariant under the action of Γ. Now in the odd case, we obtain a bounded uniformly continuous function

$$\widetilde{U} : [0, \infty) \to (C^*(\widetilde{X})^\Gamma)^+$$

such that the propagation of $\widetilde{U}_t$ goes to zero, as $t \to \infty$. So $\widetilde{U}$ determines a K-theory class in $K_1(C_L^*(\widetilde{X})^\Gamma)$. Moreover, this class coincides with the K-homology class $[D_{\mathrm{sig}}]$ in $K_1(C_L^*(X))$, after the natural identifications

$$K_1(C_L^*(\widetilde{X})^\Gamma) \cong K_1(C_L^*(X)) \cong K_1(X).$$

The even case is similar.

5.3 Assembly map

Recall that the evaluation map

$$\mathrm{ev} : C_L^*(X)^\Gamma \to C^*(X)^\Gamma$$

$$f(t) \mapsto f(0)$$

induces a homomorphism at the level of K-theory:

$$\mathrm{ev}_* : K_i(C_L^*(X)^\Gamma) \to K_i(C^*(X)^\Gamma),$$

where $i = 0, 1$. The ev_* in fact coincides with the Baum-Connes assembly map $\mu : K_i(X) \to K_i(C_r^*(\Gamma))$, with the natural identifications $K_i(C_L^*(X)^\Gamma) \cong K_i(X)$ and $K_i(C^*(X)^\Gamma) \cong K_i(C_r^*(\Gamma))$. In particular, it follows from our discussion above that

$$\mu[D_{\mathrm{sig}}] = \mathrm{sig}_\Gamma(X) \in K_n(C_r^*(\Gamma)),$$

where $n = \dim X$. Let us summarize this in the following proposition.

Proposition 5.6. *For each closed oriented Witt space of dimension n, we have*

$$\mu[D_{\mathrm{sig}}] = \mathrm{sig}_\Gamma(X) \in K_n(C_r^*(\Gamma)).$$

A Minimal elements in $W_i^{\bar{p}}(X)$

The purpose of this section is to show that it is always possible to choose a basis $\{x_\alpha\}$ of $W_i^{\bar{p}}(X)$ such that x_α are uniformly bounded in the following sense. Let $\sharp x_\alpha$ be the number of simplices in x_α. One can always choose $\{x_\alpha\}$ such that $\sharp x_\alpha$ is uniformly bounded. In particular, apply the refinement map $\mathcal{A}$ repeatedly, one can choose a basis of $\mathcal{A}W_i^{\bar{p}}(X)$ such that the supports of these basis elements shrink uniformly.

Given a pseudomanifold X, let T be a triangulation of X. Recall the construction of $W_i^{\bar{p}}$ from Section 2.3. Denote the first barycentric subdivision of T by T'. Consider the stratification of X given by the skeleton of T,

$$X = |T_n| \supset \Sigma = |T_{n-2}| \supset |T_{n-3}| \supset \cdots \supset |T_0|.$$

Define $R_i^{\bar{p}}$ to be the subcomplex of T' consisting of all simplices which are $(\bar{p}, i)$-allowable with respect to this stratification. $W_i^{\bar{p}}(X)$ is defined to be the subgroup of $C_i^{T'}(R_i^{\bar{p}})$ consisting of those simplicial i-chains with boundary supported on $R_{i-1}^{\bar{p}}$.

Denote an element of $W_k^{\bar{p}}(X)$ by a sum $\sum_{i\in\mathcal{I}} a_i\sigma_i$ for some finite index set $\mathcal{I}$ and $a_i \in \mathbb{Z}$. Here σ_i is a k-simplex of the triangulation T'. Without loss of generality, we assume that the geometric support of $\sum_{i\in\mathcal{I}} a_i\sigma_i$ in X is connected. Moreover, by reversing the orientation of σ_i if necessary, we assume that $a_i > 0$. Let σ be a summand of the sum $\sum_{i\in\mathcal{I}} a_i\sigma_i$.

Definition A.1. An element $\sum_{i\in\mathcal{I}} a_i\sigma_i$ of $W_k^{\bar{p}}(X)$ with $a_i > 0$ is called minimal if it *cannot* be written as a sum

$$\sum_{i\in\mathcal{I}} \alpha_i\sigma_i + \sum_{i\in\mathcal{I}} \beta_i\sigma_i$$

such that

(i) $\alpha_i, \beta_i \geq 0$ and $\alpha_i + \beta_i = a_i$;

(ii) both $\sum_{i\in\mathcal{I}} \alpha_i\sigma_i$ and $\sum_{i\in\mathcal{I}} \beta_i\sigma_i$ are in $W_k^{\bar{p}}(X)$.

Clearly, the support of each minimal element is connected.

Lemma A.2. *There exists a universal constant $c_k > 0$ such that $|\mathcal{I}| \leq c_k$ for all minimal elements $\sum_{i\in\mathcal{I}} a_i\sigma_i$ of $W_k^{\bar{p}}(X)$. Here c_k only depends on k.*

Proof. Choose a summand σ_e of $\sum_{i\in\mathcal{I}} a_i\sigma_i$. Observe that, up to orientation, σ_e is a simplex of the form $[v_0,\cdots v_k]$, where v_i is the barycenter of some simplex Δ_{v_i} in T and Δ_{v_i} is a sub-simplex of $\Delta_{v_{i+1}}$ for all $0\le i\le k-1$. Let us write $\tilde\sigma=[v_0,\cdots v_k]$. Then $\sigma_e=\pm\tilde\sigma$.

Claim. The face $[v_1,\cdots,v_k]$ is supported on $R^{\bar p}_{k-1}$.

Let us assume the opposite, that is, $[v_1,\cdots,v_k]$ is not supported on $R^{\bar p}_{k-1}$. This implies that there exists j such that

$$\dim([v_1,\cdots,v_k]\cap X_{n-j})>k-1-j+p_j.$$

In particular, $\dim([v_1,\cdots,v_k]\cap X_{n-j})\ge 0$ and we have that $v_s\in X_{n-j}$ for some $1\le s\le k$. Since v_s is the barycenter of Δ_{v_s}, it follows that Δ_{v_s} is contained in X_{n-j}. In particular, this implies that v_0 lies in X_{n-j} as well. Therefore, we have

$$\dim(\tilde\sigma\cap X_{n-j})=\dim([v_1,\cdots,v_k]\cap X_{n-j})+1>k-j+p_k,$$

which contradicts with the assumption that $\tilde\sigma\in R^{\bar p}_k$. This proves the claim.

Now let us prove the lemma by induction.

(1) If all the faces $[v_0,v_1,\cdots,\hat v_\ell,\cdots,v_k]$ of $\tilde\sigma$ lie in $R^{\bar p}_{k-1}$, then $\tilde\sigma$ (and equivalently σ_e) is an element of $W^{\bar p}_k(X)$. Then

$$\sum_{i\in\mathcal{I}} a_i\sigma_i=\left[\sum_{i\neq e\in\mathcal{I}} a_i\sigma_i+(a_e-1)\sigma_e\right]+\sigma_e.$$

This contradicts with the assumption that $\sum_{i\in\mathcal{I}} a_i\sigma_i$ is minimal. So at least one face, say $[v_0,v_1,\cdots,\hat v_\ell,\cdots,v_k]$ with $\ell>0$, is *not* supported on $R^{\bar p}_{k-1}$.

(2) By assumption, $\partial(\sum_{i\in\mathcal{I}} a_i\sigma_i)$ is supported on $R^{\bar p}_{k-1}$. It follows that there exists another summand $\omega\neq\sigma_e$ such that a face of ω cancels out with $[v_0,v_1,\cdots,\hat v_\ell,\cdots,v_k]$. Let us write

$$\omega=\pm[w_0,w_1,\cdots,w_k],$$

where again w_j is the barycenter of some simplex Δ_{w_j} of T and Δ_{w_j} is contained in $\Delta_{w_{j+1}}$ for all $0\le j\le k-1$. Then clearly we have $w_0=v_0$. In particular, it follows that ω and σ_e both lie in the star of the vertex v_0.

Repeat the above steps, due to minimality of $\sum_{i\in\mathcal{I}} a_i\sigma_i$, it follows that all simplices σ_i in the summation contain the vertex v_0. In particular, they are all contained in the star of v_0. Now because X has bounded geometry, the number of k-simplices in the star of a vertex is uniformly bounded by some constant, say, c_k. This finishes the proof.

□

We have the following immediate corollary.

Corollary A.3. *$W^{\bar p}_k(X)$ has a basis consisting of minimal elements, each of which is supported on the star of a vertex.*

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